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On the L_2-Discrepancy of the Sobol–Hammersley Net in Dimension 3

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With the help of Walsh series analysis we show that the symmetrized Sobol–Hammersley net in base 2 and dimension 3 has almost best possible order of L_2-discrepancy.

Key Words: digital nets; L_2-discrepancy; complexity; Walsh series analysis.

1. INTRODUCTION AND STATEMENT OF THE RESULT

For a point set $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ of points in the s-dimensional unit-cube $[0, 1)^s$, and for any $\xi = (\xi_1, \ldots, \xi_s) \in [0, 1)^s$ let $S(\xi)$ denote the number of the $\mathbf{x}_i$ contained in the box $\prod_{j=1}^{s} [0, \xi_j)$. Further we write $|\xi|$ for $\xi_1 \cdot \xi_2 \cdots \xi_s$. Then the quantity

$$L_2(\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}) := \left(\int_{[0,1)^s} \left(\frac{S(\xi)}{N} - |\xi| \right)^2 d\xi \right)^{\frac{1}{2}}$$

is called the L_2-discrepancy of the point set $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$.

Roth [13] has shown that for every dimension s there is a constant $c_s > 0$ such that for every point set $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ in $[0, 1)^s$

$$L_2(\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}) > c_s \cdot \frac{(\log N)^{\frac{s-1}{2}}}{N}$$

holds.

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In [2] Davenport proved that this result is best possible in dimension 2, in [14] Roth proved that the result is best possible in dimension 3, and in [15] Roth proved that the estimate is best possible in any dimension. Finally quite recently, Chen and Skriganov [1], gave concrete examples (not only existence results as Roth did) of point sets in arbitrary dimensions of minimal order of L_2-discrepancy.

These results obtain special relevance by a theorem of Wozniakowski [19] on the average complexity of numerical integration by low-discrepancy sequences. Wozniakowski's result, roughly speaking, essentially says the following.

For given points $\mathbf{x}_0, ..., \mathbf{x}_{N-1}$ in $[0, 1)^s$ and continuous real functions f on $[0, 1)^s$ consider the "integration error"

$$\left| \int_{[0,1)^s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{N} \sum_{k=0}^{N-1} f(\mathbf{x}_k) \right|.$$

If we take now the average of this error over all continuous functions f (with respect to the Wiener sheet measure) then this average is essentially the L_2-discrepancy of the point set $\mathbf{x}_0, ..., \mathbf{x}_{N-1}$.

This result motivates the use of point sets with small L_2-discrepancy for numerical integration purposes.

In [7] we developed a method to analyze the L_2-discrepancy of symmetrisized point sets with the help of Walsh series and we used this method to estimate the L_2-discrepancy of certain two-dimensional symmetrisized digital $(0, m, 2)$-nets in base 2.

It is the aim of this paper to estimate the L_2-discrepancy of the symmetrisation of the famous three-dimensional Sobol–Hammersley net (see [3, 8, 17]), and te show that this point set has almost optimal order of L_2-discrepancy.

The three-dimensional Sobol–Hammersley net in base 2 is given by the following setting:

DEFINITION 1.1. For given dimension $s = 1, 2$, or 3 and given $m \geqslant 1$ let $C_1, ..., C_s$ be s $m \times m$-matrices over $\mathbb{Z}_2$ with the following property: For every choice of non-negative integers $d_1, ..., d_s$ with $d_1 + \cdots + d_s = m$, the system of the first d_1 rows of C_1, together with the first d_2 rows of C_2, ..., together with the first d_s rows of C_s is linearly independent over $\mathbb{Z}_2$. We then use $C_1, ..., C_s$ to construct $N = 2^m$ points $\mathbf{x}_0, ..., \mathbf{x}_{N-1}$ in $[0, 1)^s$ in the following way.

The ith coordinate $x_n^{(i)}$ of $\mathbf{x}_n$ is obtained by representing n in base 2,

$$n = n_0 + n_1 2 + \cdots + n_{m-1} 2^{m-1},$$

by multiplying C_i with the vector

$$\vec{n} = (n_0, n_1, \ldots, n_{m-1})^T$$

of digits in $\mathbb{Z}_2$,

$$C_i \cdot \vec{n} =: (y_1, y_2, \ldots, y_m)^T,$$

and by setting

$$x_n^{(i)} := \frac{y_1}{2} + \frac{y_2}{2^2} + \cdots + \frac{y_m}{2^m}.$$

Every point set generated in this way is called digital $(0, m, s)$-net in base 2 (generated by $C_1, C_2, \ldots, C_s$).

The three-dimensional digital $(0, m, 3)$-net in base 2 generated by

$$C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$C_3 = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \cdots & \binom{m-2}{0} & \binom{m-1}{0} \\ 0 & \binom{1}{1} & \cdots & \binom{m-2}{1} & \binom{m-1}{1} \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \cdots & \binom{m-2}{m-2} & \binom{m-1}{m-2} \\ 0 & 0 & \cdots & 0 & \binom{m-1}{m-1} \end{pmatrix} \quad \text{modulo } 2$$

is the three-dimensional Sobol–Hammersley net in base 2.

Remark 1.1. Note that also Faure and Niederreiter in different settings have introduced and analyzed these point sets.

Note that the first coordinates of this point set just run through the values $k/2^m$; $k=0, \ldots, 2^m-1$, and note also that C_2, C_3 generate a $(0, m, 2)$-net in base 2.

That means that in the subsequent main result of our paper we investigate the symmetrisation of a slightly more general point set:

THEOREM 1.1. *For any given positive integer M with, say, $2^{m-1} < M \leqslant 2^m$ consider the three-dimensional point set consisting of the $N = 4M$ points,*

$$\left(\frac{n}{M}, y_n, z_n\right), \left(\frac{n}{M}, y_n, 1-z_n\right), \left(\frac{n}{M}, 1-y_n, z_n\right), \left(\frac{n}{M}, 1-y_n, 1-z_n\right);$$
$$n = 0, \ldots, M-1,$$

where

$$(y_0, z_0), (y_1, z_1), \ldots, (y_{2^m-1}, z_{2^m-1})$$

is the digital $(0, m, 2)$-net in base 2 generated by the $m \times m$-unit matrix C_2 and the $m \times m$-"Pascal matrix"

$$C_3 = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \cdots & \binom{m-2}{0} & \binom{m-1}{0} \\ 0 & \binom{1}{1} & \cdots & \binom{m-2}{1} & \binom{m-1}{1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \binom{m-2}{m-2} & \binom{m-1}{m-2} \\ 0 & 0 & \cdots & 0 & \binom{m-1}{m-1} \end{pmatrix},$$

i.e., $C_3 = (b_{i,j})_{i,j=0,\ldots,m-1}$ with $b_{i,j} = \binom{j}{i}$ in $\mathbb{Z}_2$. The point set then has L_2-discrepancy

$$L_2 \leqslant c \cdot \frac{\log N}{N} \cdot \sqrt{\log \log N}$$

with an absolute constant c not depending on N.

2. AUXILIARY RESULTS

First we repeat some facts on Walsh function systems. A reference for these facts for example is [16].

For a non-negative integer k with binary representation $k = k_0 + k_1 2 + \cdots + k_{r-1} 2^{r-1}$ the function $\mathrm{wal}_k \colon \mathbb{R} \to \mathbb{R}$, periodic with period one, is defined by

$$\mathrm{wal}_k(x) = (-1)^{k_0 x_1 + \cdots + k_{r-1} x_r}$$

when $x \in [0, 1)$ has binary representation $x = x_1/2 + x_2/2^2 + \cdots$ (unique in the sense that infinitely many of the x_i must be different from one).

The system $\{\mathrm{wal}_k \mid k = 0, 1, \dots\}$ is a complete orthonormal system in $L^2([0, 1))$.

By $\oplus$ we denote digit-wise addition modulo 2, i.e., for $x = \sum_{i=w}^{\infty} (x_i/2^i)$ and $y = \sum_{i=w}^{\infty} (y_i/2^i)$ we have

$$x \oplus y := \sum_{i=w}^{\infty} \frac{z_i}{2^i}, \qquad \text{where} \quad z_i := x_i + y_i \text{ modulo } 2.$$

We have

$$\mathrm{wal}_k(x) \cdot \mathrm{wal}_l(x) = \mathrm{wal}_{k \oplus l}(x), \qquad \mathrm{wal}_k(x) \cdot \mathrm{wal}_k(y) = \mathrm{wal}_k(x \oplus y)$$

and

$$\mathrm{wal}_1(2^l x) = \mathrm{wal}_{2^l}(x).$$

For dimension $s \geqslant 2$ and non-negative integers $k_1, k_2, \dots, k_s$ the function $\mathrm{wal}_{k_1, \dots, k_s} \colon \mathbb{R}^s \to \mathbb{R}$ is defined by

$$\mathrm{wal}_{k_1, \dots, k_s}(x_1, \dots, x_s) = \prod_{i=1}^{s} \mathrm{wal}_{k_i}(x_i).$$

The system $\{\mathrm{wal}_{k_1, \dots, k_s} \mid k_1, \dots, k_s \geqslant 0\}$ is a complete orthonormal system in $L_2([0, 1)^s)$.

The following relations easily can be checked by little calculation (or see for example, [5, 10]).

Lemma 2.1. *For $k > 0$ we have*

$$\int_0^1 \xi \cdot \mathrm{wal}_k(\xi)\, d\xi = \begin{cases} -\dfrac{1}{2^{r+2}} & \text{if} \quad k = 2^r \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.2. *Let ψ be the function periodic with period one on $\mathbb{R}$ defined by*

$$\psi(x)=\begin{cases} x & \text{if } 0\leqslant x<\frac{1}{2} \\ x-1 & \text{if } \frac{1}{2}\leqslant x<1. \end{cases}$$

Then for $k>0$ and $0\leqslant x\leqslant 1$ we have

$$\int_x^1 \mathrm{wal}_k(\xi)\, d\xi=-\mathrm{wal}_k(x)\cdot\frac{1}{2^r}\cdot\psi(2^r x),$$

where r is such that $2^r\leqslant k<2^{r+1}$.

LEMMA 2.3. *For the function ψ of Lemma 2.2 we have*

$$\psi(y)=\frac{1}{4}\,\mathrm{wal}_1(y)-\sum_{p=1}^{\infty}\frac{1}{2^{p+2}}\,\mathrm{wal}_{2^p}(y).$$

Proof. By using Lemma 2.1 and Lemma 2.2 we find that this is the Walsh series of ψ. The equality for $y=\frac{1}{2}$ is checked by direct calculation. ▮

LEMMA 2.4. *Let the non-negative integer k with binary representation*

$$k=k_0+k_1 2+\cdots+k_r 2^r, \qquad k_r=1$$

be given. Then for any $y\in[0,1]$ we have

$$\mathrm{wal}_k(1-y)\cdot\psi(2^r(1-y))=-(-1)^{\sigma(k)}\cdot\mathrm{wal}_k(y)\cdot\psi(2^r y),$$

where $\psi(k)$ is the sum of digits of k in base 2.

Proof. See [7, Lemma 4]. ▮

LEMMA 2.5. *Let the non-negative integer U have binary expansion $U=U_0+U_1 2+\cdots+U_{m-1}2^{m-1}$. For any non-negative integer $n\leqslant U-1$ let $n=n_0+n_1 2+\cdots+n_{m-1}2^{m-1}$ be the binary representation of n. For $0\leqslant p\leqslant m-1$ let $U(p):=U_0+\cdots+U_p 2^p$. Let $b_0, b_1, \ldots, b_{m-1}$ be arbitrary elements of $\mathbb{Z}_2$, not all zero. Then*

$$\sum_{n=0}^{U-1}(-1)^{b_0 n_0+\cdots+b_{m-1}n_{m-1}}$$
$$=(-1)^{b_{w+1}U_{w+1}+\cdots+b_{m-1}U_{m-1}}\cdot(2^w-1+(-1)^{U_w}(U(w)-2^w+1)),$$

where w is minimal such that $b_w = 1$. Especially we have

$$\left|\sum_{n=0}^{U-1} (-1)^{b_0 n_0 + \cdots + b_{m-1} n_{m-1}}\right| \leqslant \min(U, 2^w).$$

Proof. See [7, Lemma 5]. ∎

LEMMA 2.6. *For an $m \times m$-matrix C and some k with $1 \leqslant k \leqslant m$ let $C(k)$ be the left upper $k \times k$-submatrix of C. If the $m \times m$-matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_2$ with*

$$C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

generate a digital $(0, m, s)$-net in base 2, then for all k with $1 \leqslant k \leqslant m$ the matrices $C_2(k), \ldots, C_s(k)$ generate a digital $(0, k, s-1)$-net in base 2.

Proof. Let $d_1, \ldots, d_s$ be non-negative integers with $d_1 = m-k$ and $d_2 + \cdots + d_s = k$. Then the system formed of all the first d_i rows of the C_i for $i = 1, \ldots, s$ is linearly independent over $\mathbb{Z}_2$. Therefore, however, the system formed of the first d_i rows of $C_i(k)$ for now $i = 2, \ldots, s$ is linearly independent over $\mathbb{Z}_2$. ∎

For a given $s \geqslant 1$ and a given point set $\mathbf{x}_0, \ldots, \mathbf{x}_{M-1}$ in $[0, 1)^s$ with $\mathbf{x}_n := (x_n^{(1)}, \ldots, x_n^{(s)})$ we consider the $s+1$-dimensional point set of $N = 2^s M$ points in $[0, 1)^{s+1}$ of the form

$$\left(\frac{n}{M}, \tilde{x}_n^{(1)}, \ldots, \tilde{x}_n^{(s)}\right); \qquad n = 0, \ldots, M-1,$$

where $\tilde{x}_n^{(i)}$ runs through the two values $x_n^{(i)}$ or $1 - x_n^{(i)}$ for $i = 1, \ldots, s$. So for given n we obtain 2^s points for our point set.

Then we have

PROPOSITION 2.1. *The L_2-discrepancy of a symmetrisized point set of the $N = 2^s M$ points $(\frac{n}{M}, \tilde{x}_n^{(1)}, \ldots, \tilde{x}_n^{(s)})$; $n = 0, \ldots, M-1$ in $[0, 1)^{s+1}$ satisfies*

$$c'_s \cdot \frac{1}{M} \sum_{U=0}^{M-1} L(U) \leqslant (N L_2)^2 \leqslant c_s \cdot \frac{1}{M} \sum_{U=0}^{M-1} L(U),$$

where c_s', c_s are positive constants only depending on the dimension s, where

$$L(U)=\sum_{\substack{k_1,\ldots,k_s\geqslant 0\\(k_1,\ldots,k_s)\neq(0,\ldots,0)}}\left(\sum_{n=0}^{U-1}{}^{*}\hat{F}_{\tilde{x}_n^{(1)},\ldots,\tilde{x}_n^{(s)}}(k_1,\ldots,k_s)\right)^2,$$

and where

$$\hat{F}_{y_1,\ldots,y_s}(k_1,\ldots,k_s)=\prod_{i=1}^{s}\int_{y_i}^{1}\mathrm{wal}_{k_i}(\xi)\,d\xi-\prod_{i=1}^{s}\int_{0}^{1}\xi\cdot\mathrm{wal}_{k_i}(\xi)\,d\xi.$$

(Here by $\sum_{n=0}^{U-1*}$ we denote summation over all $n=0,\ldots,U-1$ and for every n over all 2^s combinations of the $x_n^{(i)}$.)

Proof. See [7, Proposition 1]. ∎

3. PROOF OF THEOREM 1.1: PREREQUESITES

First we have to prepare the technical tools. Using the formula for $\hat{F}_{y_1,\ldots,y_s}(k_1,\ldots,k_s)$ stated in Proposition 2.1 as well as Lemma 2.1 and Lemma 2.2 we obtain:

for $k_1>0$,

$$\hat{F}_{x,y}(k_1,0)=(y-1)\,\mathrm{wal}_{k_1}(x)\,\frac{1}{2^{r(k_1)}}\,\psi(2^{r(k_1)}x)+\begin{cases}\dfrac{1}{2^{r(k_1)+3}} & \text{if}\quad k_1=2^{r(k_1)}\\ 0 & \text{otherwise}\end{cases}$$

for $k_2>0$,

$$\hat{F}_{x,y}(0,k_2)=(x-1)\,\mathrm{wal}_{k_2}(y)\,\frac{1}{2^{r(k_2)}}\,\psi(2^{r(k_2)}y)+\begin{cases}\dfrac{1}{2^{r(k_2)+3}} & \text{if}\quad k_2=2^{r(k_2)}\\ 0 & \text{otherwise}\end{cases}$$

for $k_1, k_2 > 0$,

$$\hat{F}_{x,y}(k_1, k_2) = \mathrm{wal}_{k_1}(x)\,\mathrm{wal}_{k_2}(y)\,\frac{1}{2^{r(k_1)+r(k_2)}}\,\psi(2^{r(k_1)}x)\,\psi(2^{r(k_2)}y)$$

$$-\begin{cases} \dfrac{1}{2^{r(k_1)+r(k_2)+4}} & \text{if } \; k_1 = 2^{r(k_1)} \; \text{ and } \; k_2 = 2^{r(k_2)} \\ 0 & \text{otherwise.} \end{cases}$$

We will show, that with an absolute constant c, for all m and all U with $0 \leqslant U \leqslant 2^m - 1$, for the special point set $(\tilde{y}_n, \tilde{z}_n)$, $n = 0, \ldots, U-1$, where $\tilde{y}_n$ runs through the two values y_n and $1 - y_n$ and $\tilde{z}_n$ runs through the two values z_n and $1 - z_n$, we have

$$L(U) = \sum_{\substack{k_1, k_2 \geqslant 0 \\ (k_1, k_2) \neq (0,0)}} \left(\sum_{n=0}^{U-1}{}^{*}\, \hat{F}_{\tilde{y}_n, \tilde{z}_n}(k_1, k_2) \right)^2 \leqslant c \cdot m^2 \cdot \log m.$$

By Proposition 2.1 and since $m \leqslant \frac{\log N}{\log 2} - 1$ the result then follows.

The rather boring technical but simple details from here on until the end of this section are intended to remove inessential parts of $L(U)$ and to show that we can restrict to estimate the essential part $\sum$ (for the definition of see $\sum$ see Section 4) of $L(U)$ which is done in the final two sections.

Fortunately by Lemma 2.4 we get

$$\sum_{n=0}^{U-1}{}^{*}\, \hat{F}_{\tilde{y}_n, \tilde{z}_n}(k_1, 0) = \begin{cases} 0 & \text{if } \; \sigma(k_1) \text{ is even} \\ \displaystyle\sum_{n=0}^{U-1} 2\hat{F}_{\tilde{y}_n}(k_1) & \text{otherwise} \end{cases}$$

and

$$\sum_{n=0}^{U-1}{}^{*}\, \hat{F}_{\tilde{y}_n, \tilde{z}_n}(0, k_2) = \begin{cases} 0 & \text{if } \; \sigma(k_2) \text{ is even} \\ \displaystyle\sum_{n=0}^{U-1} 2\hat{F}_{\tilde{z}_n}(k_2) & \text{otherwise.} \end{cases}$$

So the parts of $L(U)$ with $k_1 = 0$ or $k_2 = 0$ can be estimated like in [7, Theorem 2] (note that C_2 and C_3 are right upper-triangular matrices). Further by Lemma 2.4 for $k_1, k_2 > 0$ we have

$$\sum_{n=0}^{U-1}{}^{*}\, \hat{F}_{\tilde{y}_n, \tilde{z}_n}(k_1, k_2) = (1 - (-1)^{\sigma(k_1)})(1 - (-1)^{\sigma(k_2)}) \sum_{n=0}^{U-1} \hat{F}_{y_n, z_n}(k_1, k_2)$$

so that we just have to estimate

$$\tilde{L}(U) := 16 \sum_{\substack{k_1, k_2 > 0 \\ \sigma(k_1), \sigma(k_2) \text{ odd}}} \left(\sum_{n=0}^{U-1} \hat{F}_{y_n, z_n}(k_1, k_2) \right)^2.$$

We consider now the case where k_1 and k_2 are powers of 2. Inserting for $\hat{F}_{y_n, z_n}$ using Lemma 2.3 and a little calculation yields

$$\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \left(\sum_{n=0}^{U-1} \hat{F}_{y_n, z_n}(2^{r_1}, 2^{r_2}) \right)^2$$

$$= \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \frac{1}{4^{r_1+r_2}}$$

$$\times \left(\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{n=0}^{U-1} \frac{1}{2^{p+q+4}} \operatorname{wal}_{2^{r_1}}(y_n) \operatorname{wal}_{2^{r_1+p}}(y_n) \operatorname{wal}_{2^{r_2}}(z_n) \operatorname{wal}_{2^{r_2+q}}(z_n) \right.$$

$$- \sum_{q=1}^{\infty} \frac{1}{2^{q+4}} \sum_{n=0}^{U-1} \operatorname{wal}_{2^{r_2}}(z_n) \operatorname{wal}_{2^{r_2+q}}(z_n)$$

$$\left. - \sum_{p=1}^{\infty} \frac{1}{2^{p+4}} \sum_{n=0}^{U-1} \operatorname{wal}_{2^{r_1}}(y_n) \operatorname{wal}_{2^{r_1+p}}(y_n) \right)^2.$$

The second and the third term in the brackets are estimated as they were in the proof of Theorem 2 in [7] (note again that both $y_0, \ldots, y_{U-1}$ and $z_0, \ldots, z_{U-1}$ are generated by right upper-triangular matrices). So we only have to be concerned with the first term.

Now for $k_1, k_2 > 0$, $(k_1, k_2) \neq (2^{r(k_1)}, 2^{r(k_2)})$, inserting for $\hat{F}_{y_n, z_n}$ and Lemma 2.3 yield (we write $\grave{\sum}$ for $\sum_{\substack{k_1, k_2 > 0, \\ (k_1, k_2) \neq (2^{r(k_1)}, 2^{r(k_2)})}}$)

$$\grave{\sum} \left(\sum_{n=0}^{U-1} \hat{F}_{y_n, z_n}(k_1, k_2) \right)^2$$

$$\leqslant 4 \grave{\sum} \frac{1}{4^{r(k_1)+r(k_2)}}$$

$$\times \left(\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{2^{p+q+4}} \left| \sum_{n=0}^{U-1} \operatorname{wal}_{k_1+2^{p+r(k_1)}}(y_n) \operatorname{wal}_{k_2+2^{q+r(k_2)}}(z_n) \right| \right)^2$$

$$+ 4 \grave{\sum} \frac{1}{4^{r(k_1)+r(k_2)}} \left(\sum_{p=1}^{\infty} \frac{1}{2^{p+4}} \left| \sum_{n=0}^{U-1} \operatorname{wal}_{k_1+2^{p+r(k_1)}}(y_n) \operatorname{wal}_{k_2 \oplus 2^{r(k_2)}}(z_n) \right| \right)^2$$

$$+ 4 \grave{\sum} \frac{1}{4^{r(k_1)+r(k_2)}} \left(\sum_{q=1}^{\infty} \frac{1}{2^{q+4}} \left| \sum_{n=0}^{U-1} \operatorname{wal}_{k_1 \oplus 2^{r(k_1)}}(y_n) \operatorname{wal}_{k_2+2^{q+r(k_2)}}(z_n) \right| \right)^2$$

$$+ 4 \grave{\sum} \frac{1}{4^{r(k_1)+r(k_2)}} \left(\sum_{n=0}^{U-1} \frac{1}{2^4} \operatorname{wal}_{k_1 \oplus 2^{r(k_1)}}(y_n) \operatorname{wal}_{k_2 \oplus 2^{r(k_2)}}(z_n) \right)^2.$$

We show now that there is no need to investigate the last three sums separately. Take for example

$$\grave{\sum} \frac{1}{4^{r(k_1)+r(k_2)}} \left(\sum_{q=1}^{\infty} \frac{1}{2^{q+4}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{k_1 \oplus 2^{r(k_1)}}(y_n)\, \mathrm{wal}_{k_2+2^{q+r(k_2)}}(z_n) \right| \right)^2.$$

Since the sum over those k_1 which equal $2^{r(k_1)}$, because of $k_1 \oplus 2^{r(k_1)} = 0$ again reduces to the sum already estimated in the proof of Theorem 2 in [7], it suffices to estimate (later on we will set $k_1 = \rho_1 + 2^{\kappa + r(\rho_1)}$ with $\kappa \geqslant 1$, $\rho_1 = \tau_1 + 2^{r(\rho_1)}$ with $\tau_1 < 2^{r(\rho_1)}$, and we will use the convention $r(0) := 0$)

$$\begin{aligned}
&\sum_{\substack{k_1, k_2 > 0 \\ k_1 \neq 2^{r(k_1)}}} \frac{1}{4^{r(k_1)+r(k_2)}} \\
&\qquad \times \left(\sum_{q=1}^{\infty} \frac{1}{2^{q+4}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{k_1 \oplus 2^{r(k_1)}}(y_n)\, \mathrm{wal}_{k_2+2^{q+r(k_2)}}(z_n) \right| \right)^2 \\
&= \sum_{\rho_1, k_2 > 0} \frac{1}{4^{r(k_2)+r(\rho_1)}} \\
&\qquad \times \sum_{\kappa=1}^{\infty} \frac{1}{4^{\kappa}} \left(\sum_{q=1}^{\infty} \frac{1}{2^{q+4}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{\rho_1}(y_n)\, \mathrm{wal}_{k_2+2^{r(k_2)+q}}(z_n) \right| \right)^2 \\
&\leqslant \sum_{\tau_1=0}^{\infty} \sum_{k_2>0} \frac{1}{4^{r(k_2)+r(\tau_1)}} \\
&\qquad \times \sum_{\sigma=1}^{\infty} \frac{1}{4^{\sigma}} \left(\sum_{q=1}^{\infty} \frac{1}{2^{q+4}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{\tau_1+2^{r(\tau_1)+\sigma}}(y_n)\, \mathrm{wal}_{k_2+2^{r(k_2)+q}}(z_n) \right| \right)^2 \\
&\leqslant \sum_{\tau_1=0}^{\infty} \sum_{k_2>0} \frac{1}{4^{r(k_2)+r(\tau_1)}} \\
&\qquad \times \left(\sum_{p=1}^{\infty} \frac{1}{2^{p}} \sum_{q=1}^{\infty} \frac{1}{2^{q}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{\tau_1+2^{r(\tau_1)+p}}(y_n)\, \mathrm{wal}_{k_2+2^{r(k_2)+q}}(z_n) \right| \right)^2.
\end{aligned}$$

In a quite analogous way the second and the fourth term are transformed so that we altogether just have to estimate

$$\sum_{(k_1, k_2) \neq (0,0)} \frac{1}{4^{r(k_1)+r(k_2)}} \left(\sum_{p,q=1}^{\infty} \frac{1}{2^{p+q}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{k_1+2^{p+r(k_1)}}(y_n)\, \mathrm{wal}_{k_2+2^{q+r(k_2)}}(z_n) \right| \right)^2.$$

Next we show that we can restrict in the above sum to $k_1, k_2 < 2^m$. First we have

$$\sum_{k_1, k_2 \geqslant 2^m} \leqslant U^2 \cdot \sum_{k_1, k_2 \geqslant 2^{m-1}} \frac{1}{4^{r(k_1)+r(k_2)}}$$
$$\leqslant U^2 \cdot \sum_{i=m-1}^{\infty} \sum_{j=m-1}^{\infty} 2^{i+1} \cdot 2^{j+1} \cdot \frac{1}{4^{i+j}} \leqslant 64.$$

Further (we will write $k_2 = l + \lambda \cdot 2^m$ later on)

$$\sum_{k_1=0}^{2^m-1} \sum_{k_2 \geqslant 2^m}$$
$$\leqslant \sum_{k_1=0}^{2^m-1} \sum_{\lambda=1}^{\infty} \sum_{l=0}^{2^m-1} \frac{1}{4^{r(k_1)+m+[\mathrm{ld}(\lambda)]}} \left(\sum_{p=1}^{\infty} \frac{1}{2^p} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{k_2+2^{p+r(k_1)}}(y_n)\, \mathrm{wal}_l(z_n) \right| \right)^2$$
$$\leqslant 8 \left(\frac{1}{4^m} U^2 + \sum_{\substack{k_1, l=0 \\ (k_1, l) \neq (0,0)}}^{2^m-1} \frac{1}{4^{r(k_1)+r(l)}} \right.$$
$$\left. \times \left(\sum_{p=1}^{\infty} \frac{1}{2^p} \cdot \frac{1}{2^{m-r(l)}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{k_1+2^{p+r(k_1)}}(y_n)\, \mathrm{wal}_{l+2^{r(l)+(m-r(l))}}(z_n) \right| \right)^2 \right)$$
$$\leqslant 8 + 8 \cdot \sum,$$

where $\sum$ is the essential sum, defined below.

$\sum_{k_1 \leqslant 2^m} \sum_{k_2=0}^{2^m-1}$ is estimated in the same way, so that it suffices to estimate $\sum$.

4. PROOF OF THEOREM 1.1: GENERAL ESTIMATES FOR THE ESSENTIAL SUM

We have to show that for some absolute constant c we have

$$\sum := \sum_{\substack{k, l=0 \\ (k, l) \neq (0,0)}}^{2^m-1} \frac{1}{4^{r+s}} \cdot \left(\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{2^{p+q}} \left| \sum_{n=0}^{U-1} \mathrm{wal}_{k+2^{p+r}}(y_n)\, \mathrm{wal}_{l+2^{q+s}}(z_n) \right| \right)^2$$
$$\leqslant c \cdot m^2 \cdot \log m.$$

Here we use the notation $r := r(k)$, and $s := r(l)$.

In this section we just will use the $(0, m, s)$-net property of the underlying point set, but not special properties of the Pascal matrix. A deeper insight into the structure of the Pascal matrix then will be the central point of the last section.

We assume in the following:

$r < s$ ($r \geqslant s$ is treated in absolutely the same way)

$s < r+p$ ($s \geqslant r+p$ is treated in absolutely the same way, indeed it is much easier to deal with, since the "most critical case" $r+p = s+q$ then cannot occur)

$r+p \leqslant s+q$ ($r+p > s+q$ is treated in absolutely the same way).

We consider

Case A. $\quad r+p < s+q.$

Case B. $\quad r+p < s+q.$

With

$$k = k_0 + k_1 2 + \cdots + k_r 2^r$$

$$l = l_0 + l_1 2 + \cdots + l_s 2^s$$

$$n = n_0 + n_1 2 + \cdots + n_{m-1} 2^{m-1}$$

($k_r = l_s = 1$) we have

$$\mathrm{wal}_{k+2^{p+r}}(y_n)\,\mathrm{wal}_{l+2^{q+s}}(z_n) = (-1)^{\zeta_0 n_0 + \cdots + \zeta_{m-1} n_{m-1}},$$

where

in Case A,

$$\zeta_i = k_i + \sum_{j=0}^{i-1} l_j b_{j,\,i} + l_i \qquad \text{for} \quad i = 0, ..., r$$

$$\zeta_i = \sum_{j=0}^{i-1} l_j b_{j,\,i} + l_i \qquad \text{for} \quad i = r+1, ..., s$$

$$\zeta_i = \sum_{j=0}^{s} l_j b_{j,\,i} \qquad \text{for} \quad i = s+1, ..., p+r-1$$

$$\zeta_{p+r} = \sum_{j=0}^{s} l_j b_{j,\,p+r} + 1$$

$$\zeta_i = \sum_{j=0}^{s} l_j b_{j,\,i} \qquad \text{for} \quad i = p+r+1, ..., s+q-1$$

$$\zeta_{s+q} = \sum_{j=0}^{s} l_j b_{j,\,s+q} + 1$$

$$\zeta_i = \sum_{j=0}^{s} l_j b_{j,\,i} + b_{s+q,\,i} \qquad \text{for} \quad i = s+q+1, ..., m-1.$$

in Case B, ζ_i is for $i=0, \ldots, p+r-1$ and $i=s+q+1, \ldots, m-1$ like in case A. Since $p+r=s+q$ then there only remains the index $i=p+r$:

$$\zeta_{p+r} = \sum_{j=0}^{s} l_j b_{j,\, p+r}.$$

In the following we collect some useful information on the above system of equations:

(1) In both cases, for given r and s, for a given k with $r(k)=r$, and a given w with $0 \leqslant w \leqslant s$, there are at most 2^{s-w} values l with $r(l)=s$, and $(k, l) \neq (0, 0)$ such that $\zeta_0 = \cdots = \zeta_w = 0$.

(2) In both cases, for given r, s, p, q and w with $s+1 \leqslant w \leqslant \min(m-1, r+s+1)$ there are at most $2^{r+s+1-w}$ pairs k and l with $r(k)=r$, $r(l)=s$ and $(k, l) \neq (0, 0)$ such that $\zeta_0 = \cdots = \zeta_w = 0$.

For this fact we used the $(0, m, s)$-net property of the underlying point set: note, that by Lemma 2.6 the coefficient vectors of $l_0, \ldots, l_{s-1}, k_0, \ldots, k_{w-s-1}$ of dimension $w+1$ in the system $\zeta_0 = \cdots = \zeta_w = 0$ are linearly independent, and so the system has rank $w+1$.

(3) Let especially $w=r+s+1$ (in case that $r+s+1 \leqslant m-1$). Then in both cases the system $\zeta_0 = \cdots = \zeta_w = 0$ is homogeneous if $r+p > w$, i.e., if $p > s+1$. In this case the only solution of the system is the nonadmissible pair $(k, l) = (0, 0)$. So the system has admissible solutions only for $p \leqslant s+1$.

We consider first the part $\sum_1$ of $\sum$ over those k and l for which we do not have $\zeta_0 = \cdots = \zeta_s = 0$. For these k and l we define $w(k, l) := \min\{w \mid \zeta_w = 1\}$ (note that $w(k, l)$ indeed does not depend on p and q). Then by using Lemma 2.5 and property (1) we have

$$\begin{aligned}
\Sigma_1 &\leqslant \sum_{\substack{k, l=0 \\ (k, l) \neq (0, 0) \\ \exists w \leqslant s \text{ with } \zeta_w = 1}}^{2^m - 1} \frac{1}{4^{r+s}} \cdot 2^{2w(k, l)} \\
&\leqslant \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \sum_{w=0}^{s} \sum_{\substack{k, l \\ r(k)=r \\ r(l)=s \\ w(k, l)=w}} 2^{2w} \\
&\leqslant 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \sum_{w=0}^{s} 2^{w-r-s} \leqslant 4m.
\end{aligned}$$

In the remaining part $\sum_2$, for given r, s and k with $r(k)=r$ the maximal one l with $r(l)=s$ and $\zeta_0 = \cdots = \zeta_s = 0$ always automatically will be chosen.

For given s, k, p and q (r and l then are determined) let now

$$w(s, k, p, q) := \min\{m, \min\{w \mid \zeta_w = 1\}\}.$$

In the following, if not necessary, the dependence of w on some of the parameters s, k, p, q is not expressed explicitly, or sometimes also incompletely.

Two remarks on w will be useful:

(4) By property (3) from above we obtain that for $(k, l) \neq (0, 0)$ we can have $w(s, k, p, q) > r+s+1$ only if $p \leqslant s+1$.

(5) We will need a general estimate for w, valid in any case. Such a general estimate will be given in

LEMMA 4.1. *We always have*

$$w(s, k, p, q) \leqslant p+q+r+s+1.$$

Proof. Assume that for some $M > p+q+r+s+1$ and some $(k, l) \neq (0, 0)$ we had $\zeta_0 = \cdots = \zeta_M = 0$. Then the vectors

$$(\delta_{e,0}, \ldots, \delta_{e,M}); \qquad e = 0, \ldots, p+r,$$

where $\delta_{i,j}$ is the Kronecker symbol, together with

$$(b_{f,0}, \ldots, b_{f,M}); \qquad f = 0, \ldots, q+s$$

were linearly dependent, which contradicts Lemma 2.6. ▮

We proceed with $\sum_2$ and obtain (note that $p+r > s$, hence $p \geqslant s-r+1$ and $q+s \geqslant p+r$, hence $q \geqslant p+r-s$)

$$\begin{aligned}
\Sigma_2 \leqslant\; & 4 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \sum_{k=2^r}^{2^{r+1}-1} \Bigg(\sum_{p=s-r+1}^{s} \sum_{\substack{q=p+r-s\\ w \leqslant r+p}}^{\infty} \frac{2^w}{2^{p+q}} \Bigg)^2 \\
& +4 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \sum_{k=2^r}^{2^{r+1}-1} \Bigg(\sum_{p=s+1}^{\infty} \sum_{\substack{q=p+r-s\\ w \leqslant r+p}}^{\infty} \frac{2^w}{2^{p+q}} \Bigg)^2 \\
& +4 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \sum_{k=2^r}^{2^{r+1}-1} \Bigg(\sum_{p=s-r+1}^{\infty} \sum_{\substack{q=p+r-s\\ r+p < w \leqslant r+s+1}}^{\infty} \frac{2^w}{2^{p+q}} \Bigg)^2 \\
& +4 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \sum_{k=2^r}^{2^{r+1}-1} \Bigg(\sum_{p=s-r+1}^{s+1} \sum_{\substack{q=p+r-s\\ w \geqslant r+s+2}}^{\infty} \frac{2^w}{2^{p+q}} \Bigg)^2 \\
=:\; & \Sigma_{21} + \Sigma_{22} + \Sigma_{23} + \Sigma_{24}.
\end{aligned}$$

Now we have (with absolute constant c)

$$\Sigma_{21} \leqslant c,$$

trivially, and

$$\Sigma_{22} \leqslant 4 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \sum_{k=2^r}^{2^{r+1}-1} \left(\sum_{p=s+1}^{\infty} \sum_{q=p+r-s}^{\infty} \frac{2^{r+s+1}}{2^{p+q}} \right)^2 \leqslant c,$$

by using property (4) from above. By using property (2) we obtain

$$\Sigma_{23} \leqslant 4 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \sum_{k=2^r}^{2^{r+1}-1} (r+1) \sum_{w=s+1}^{r+s+1} \left(\sum_{p=s-r+1}^{\infty} \sum_{\substack{q=p+r-s \\ w(k,p,q)=w}}^{\infty} \frac{2^w}{2^{p+q}} \right)^2 \leqslant c \cdot m^2.$$

It remains to consider Σ_{24}, i.e., k and l with $w(k, l, p, q) \geqslant r+s+2$ for some p and q. Note that by property (2) for r, s, p, q given, there is at most one pair $(k, l) \neq (0, 0)$ with $r(k) = r$, $r(l) = s$ and $w(k, l, p, q) \geqslant r+s+2$. If for given r, s, p, q such k and l exist, then we denote by $\bar{w} := \bar{w}(r, s, p, q) \geqslant r+s+2$ the corresponding minimal value w for which $\zeta_w = 1$ (or $\bar{w} = m$ if $\zeta_w = 0$ for all w). Then

$$\begin{aligned}
\Sigma_{24} &\leqslant 4 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \left(\sum_{p=s-r+1}^{s+1} \sum_{q=p+r-s}^{\infty} \frac{2^{\bar{w}(r,s,p,q)}}{2^{p+q}} \right)^2 \\
&\leqslant 8 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \left(\sum_{p=s-r+1}^{s+1} \sum_{\substack{q=p+r-s \\ \bar{w} \leqslant p+r+s}}^{\infty} \frac{2^{\bar{w}}}{2^{p+q}} \right)^2 \\
&\quad + 8 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \left(\sum_{p=s-r+1}^{s+1} \sum_{\substack{q=p+r-s \\ \bar{w} > p+r+s}}^{\infty} \frac{2^{\bar{w}}}{2^{p+q}} \right)^2 \\
&\leqslant c \cdot m^2 + 8 \cdot \bar{\Sigma},
\end{aligned}$$

where $\bar{\Sigma}$ is the second multiple sum in the last expression. We simplify $\bar{\Sigma}$ by transforming it to

$$\bar{\Sigma} = \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \left(\sum_{q=1}^{\infty} \sum_{\substack{p=s-r+1 \\ \bar{w} > p+r+s}}^{\min(s+1,\, q+s-r)} \frac{2^{\bar{w}}}{2^{p+q}} \right)^2$$

and by noting that, because of $\bar{w} \leqslant m$, we have

$$\sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \left(\sum_{q=m+1}^{\infty} \sum_{p=s-r+1}^{\infty} \frac{2^{\bar{w}}}{2^{p+q}} \right)^2 \leqslant c \cdot m,$$

with an absolute constant c.

5. PROOF OF THEOREM 1.1: USING THE PASCAL MATRIX

So it remains to show

$$\sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \frac{1}{4^{r+s}} \left(\sum_{q=1}^{m} \sum_{\substack{p=s-r+1 \\ \bar{w} > p+r+s}}^{\min(s+1,\, q+s-r)} \frac{2^{\bar{w}}}{2^{p+q}} \right)^2 \leqslant c \cdot m^2 \cdot \log m,$$

with an absolute constant c.

From now on we will essentially use the properties of the Pascal matrix generating the third coordinate of the point set.

We consider r, s, p, q such that $\bar{w}(r, s, p, q) > p+r+s$, where $\bar{w}$ is the minimal value w such that $\zeta_w = 1$ (if it exists, otherwise it is defined by m).

From the system of equations defining $\zeta_0, \ldots, \zeta_{m-1}$ we deduce that $\bar{w}$ is therefore the maximal value w, such that a linear combination of the rows of the subsequent $(r+s+4) \times w$-matrix M, where the last two rows are weighted by 1, is the zero vector,

$$M := (A \mid B)$$

where A is the matrix

$$\begin{pmatrix}
1 & & 0 & 0 & \cdots & 0 \\
 & \ddots & & \vdots & & \vdots \\
0 & & 1 & 0 & \cdots & 0 \\
b_{0,0} & \cdots & & b_{0,s} & \cdots & b_{0,p+r-1} \\
 & \ddots & & \vdots & & \vdots \\
 & 0 & & b_{s,s} & \cdots & b_{s,p+r-1} \\
0 & \cdots & & & \cdots & 0 \\
0 & \cdots & & & \cdots & 0
\end{pmatrix},$$

and where B is the matrix

$$\left(\begin{array}{cccccc} 0 & \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & 0 \\ \vdots & \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & \vdots \\ 0 & \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & 0 \\ \hline b_{0,\,p+r} & \multicolumn{2}{c}{\cdots\cdots} & b_{0,\,q+s} & \cdots\ \cdots\ \cdots & b_{0,\,w-1} \\ \vdots & \multicolumn{2}{c}{\cdots\cdots} & \vdots & \cdots\cdots & \vdots \\ b_{s,\,p+r} & \multicolumn{2}{c}{\cdots\cdots} & b_{s,\,q+s} & \cdots\cdots & b_{s,\,w-1} \\ 0 & 0 & \cdots\ 0 & b_{q+s,\,q+s} & \cdots\cdots & b_{q+s,\,w-1} \\ 1 & 0 & \cdots & 0 & \cdots\cdots & 0 \end{array}\right).$$

Of course, the first $(r+1)$ columns and rows can be cancelled, and so $\bar{w}$ is the maximal value w such that a linear combination of the rows of the subsequent $(s+3)\times(w-r-1)$-matrix $\tilde{M}$, where the last two rows are weighted by 1, is the zero vector,

$$\tilde{M} := (C \mid D),$$

where C is the matrix

$$\begin{pmatrix} b_{0,\,r+1} & \cdots\cdots & b_{0,\,s} & \cdots & b_{0,\,p+r-1} \\ \vdots & \cdots\cdots & \vdots & \cdots & \vdots \\ b_{r+1,\,r+1} & \cdots\cdots & b_{r+1,\,s} & \cdots & b_{r+1,\,p+r-1} \\ & \ddots & \vdots & & \vdots \\ & 0 & b_{s,\,s} & \cdots & b_{s,\,p+r-1} \\ 0 & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & 0 \\ 0 & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & 0 \end{pmatrix},$$

and D is the lower part of B.

Now we use the fact that $b_{i,j} = \binom{j}{i}$ and that $p \leqslant s+1$. Then by successively subtracting (from the right to the left) the $(i-1)$st column from the ith column for $i = w-r-1, w-r-2, \ldots, u$; $u = 2, \ldots, w-r-1$, we arrive at a matrix with $s+3$ rows of the form

$$\left(\begin{array}{c|ccccc} E & \multicolumn{5}{c}{0} \\ \hline & \binom{r+1}{q-1} & \binom{r+1}{q-2} & \cdots & \cdots & \binom{r+1}{q+r+s-w+2} \\ F & & & & & \\ & \binom{s+1}{p-1} & \binom{s+2}{p-1} & \cdots & \cdots & \binom{s+(w-r-s-2)}{p-1} \end{array}\right),$$

where E is the following $(s+1)\times(s+1)$-band matrix

$$\begin{pmatrix}
\binom{r+1}{0} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\
\binom{r+1}{1} & \binom{r+1}{0} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\
\vdots & \binom{r+1}{1} & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots\\
\binom{r+1}{r+1} & \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots\\
0 & \binom{r+1}{r+1} & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\
0 & \cdots & 0 & \binom{r+1}{r+1} & \cdots & \cdots & \binom{r+1}{1} & \binom{r+1}{0} & 0\\
0 & \cdots & \cdots & 0 & \binom{r+1}{r+1} & \cdots & \cdots & \binom{r+1}{1} & \binom{r+1}{0}
\end{pmatrix}.$$

The form of the submatrix F is of no relevance.

Consequently $\bar{w}$ is the maximal value w such that the two vectors

$$\left(\binom{r+1}{q-1}, \binom{r+1}{q-2}, \ldots, \binom{r+1}{q-(w-r-s-2)}\right)$$

and

$$\left(\binom{s+1}{p-1}, \binom{s+2}{p-1}, \ldots, \binom{s+(w-r-s-2)}{p-1}\right)$$

are equal modulo 2, and so the proof of the following proposition will finish the proof of Theorem 1.1

Proposition 5.1. *For positive integers r, s, p, q let $L := L(r, s, p, q)$ be the maximal integer such that the two vectors*

$$\left(\binom{r+1}{q-1}, \binom{r+1}{q-2}, \ldots, \binom{r+1}{q-L}\right)$$

and

$$\left(\binom{s+1}{p-1}, \binom{s+2}{p-1}, \ldots, \binom{s+L}{p-1}\right)$$

are equal modulo 2. (For integers $t>q$ define $\binom{r+1}{q-t}:=0$.) Then with an absolute constant c we have for all $m\in\mathbb{N}$,

$$\bar{\sum}:=\sum_{r=1}^{m-2}\sum_{s=r+1}^{m-1}\left(\sum_{q=1}^{m}\sum_{p=s-r+1}^{s-r+q}\frac{2^L}{2^{p+q}}\right)^2\leqslant c\cdot m^2\cdot\log m.$$

Remark 5.1. Note that by using for L the estimate $L=\bar{w}-r-s-2<p+q$ which is obtained from Lemma 4.1, we only deduce the estimate $\bar{\sum}\leqslant c\cdot m^6$.

Proof of Proposition 5.1. The proof of the proposition heavily depends on the self-similar structure of the Pascal matrix (let us call it P).

It is obtained by starting with the 1×1-matrix $C_0:=(1)$.

In the $2^{n-1}\times 2^{n-1}$-matrix C_{n-1} substitute 1 by $\binom{1\ 1}{0\ 1}$ and 0 by $\binom{0\ 0}{0\ 0}$. This gives the $2^n\times 2^n$-matrix C_n. This C_n, for $n\geqslant 1$ then always is of the form

$$\begin{pmatrix}\ddots & C\\ 0 & \ddots\end{pmatrix}$$

where the boundary of the right upper triangle matrix C, including the main diagonal, consists of ones. C_n always is the left upper $2^n\times 2^n$-part of P.

Especially the left upper part of P always is of the form

$$\left(\begin{array}{cc|cc||cc|cc}
\ddots & C & \ddots & C & \ddots & C & \ddots & C\\
0 & \ddots & 0 & \ddots & 0 & \ddots & 0 & \ddots\\
\hline
\multicolumn{2}{c|}{0} & \ddots & C & \multicolumn{2}{c|}{0} & \ddots & C\\
 & & 0 & \ddots & & & 0 & \ddots\\
\hline\hline
 & & & & \ddots & C & \ddots & C\\
 & & & & 0 & \ddots & 0 & \ddots\\
\cline{5-8}
\multicolumn{4}{c||}{0} & \multicolumn{2}{c|}{0} & \ddots & C\\
 & & & & & & 0 & \ddots
\end{array}\right).$$

Here

$$\begin{pmatrix}\ddots & C\\ 0 & \ddots\end{pmatrix}$$

always is a matrix C_n for some arbitrary $n\geqslant 1$.

For given r, q we denote by S the string

$$S = \left(\binom{r+1}{q-1}, \ldots, \binom{r+1}{0} \right)$$

and for given s, p we denote by S_p the string

$$S_p = \left(\binom{s+1}{p-1}, \ldots, \binom{s+q}{p-1} \right)$$

of zeros and ones. In all what follows, for given q let l be such that

$$2^l \leqslant q-1 < 2^{l+1}.$$

That is, the vertical string S is located in the Pascal matrix as follows: the "starting point" $\binom{r+1}{q-1}$ is located anywhere in the lower half of Fig. 1 (here

$$\begin{pmatrix} \ddots & C \\ 0 & \ddots \end{pmatrix}$$

always is the matrix C_{l-1}.)

We assign to S a so-called characteristic string (C_S), i.e., a 3-string or 4-string of zeros and ones in the following way: S touches three or four $2^{l-1} \times 2^{l-1}$-matrices, either a C_{l-1} or a zero matrix (we call such matrices in P $(l-1)$-blocks). C_{l-1} obtains the value 1 and the zero matrix the value 0. So, for example, S in Fig. 1 obtains the string $C_S = 0101$.

The horizontal string S_p is of the same length as S and we also assign, in the same way a string C_{S_p} to S_p, which may be of length 3, 4, but also of length 5. (See Fig. 1 where $C_{S_{p(1)}} = 0\,0\,1\,1\,0$ and $C_{S_{p(2)}} = 0\,0\,1$.)

We sometimes have to distinguish two cases:

Case a. S and S_p have the same segmentation. That means S_p (like S) ends with the end of a $(l-1)$-block. (See $S_{p(2)}$ in Fig. 1.)

Case b. S and S_p do not have the same segmentation. (See $S_{p(1)}$ in Fig. 1.)

In the following we give estimates to which extent the strings S and S_p can have equal initial parts.

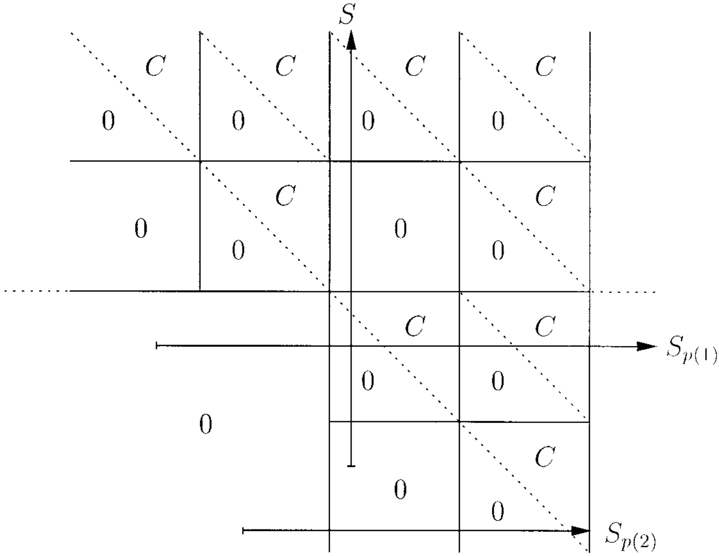

FIGURE 1

LEMMA 5.1. (a) *For given r, s, p, q let M be maximal such that the first M entries of S and S_p are equal. Then we have that all but at most one of these values $p \leqslant 2^{l-1}$ we have*

$$M < q - 2^{l-1}.$$

(b) *For given r, q, and s, and any given possible characteristic string C for S_p, at most one of the 2^{l-1} possible different strings S_p with $C_{S_p} = C$ satisfies $S_p = S$.*

Proof. Consider Case a.

Assume that $C_S \neq C_{S_p}$. Then it is obvious (see Fig. 2) that S and S_p must differ at the latest when S or S_p reaches the first entry 1 in the one of the first two corresponding different $(l-1)$-blocks which is labelled by 1.

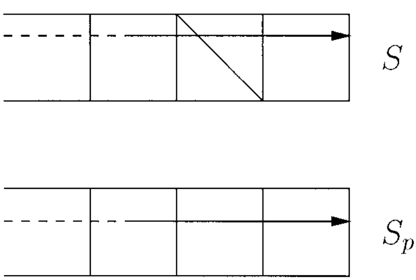

FIGURE 2

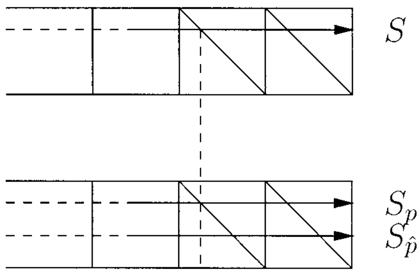

FIGURE 3

If S and S_p have the same characteristic string C, then again it is obvious (see Fig. 3), that from all possible different S_p with $C_{S_p} = C$, exactly one is equal to S. All other $S_{\hat{p}}$ differ from S in the first 1 occurring in S or in S_p.

Especially if the last three entries of C_{S_p} are ones, then for all (apart the special one) of the S_p the value M is at most $q - 2^{l-1}$.

Consider Case b. If the last three entries of the characteristic string of S_p are ones, then it again is obvious (see Fig. 4) that S and S_p differ at the latest at the first 1 before the first entry of the last $(l-1)$-block containing S.

So in this case for all S_p the value M is at most $q - 2^{l-1}$. Since the last entry of S always is 1, it is also obvious, that in any case for nonequal segmentation at most one of the different lines S_p with the given characteristic string equals to S (see Fig. 5). ▮

In the following we consider strings S and S_p with $S = S_p$, i.e., $M = q$ and $L \geqslant M$. Let $u := L - M$ this means

$$\left(\binom{r+1}{q-1}, \ldots, \binom{r+1}{0}\right) = \left(\binom{s+1}{p-1}, \ldots, \binom{s+q}{p-1}\right)$$

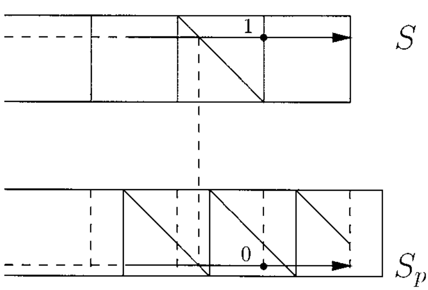

FIGURE 4

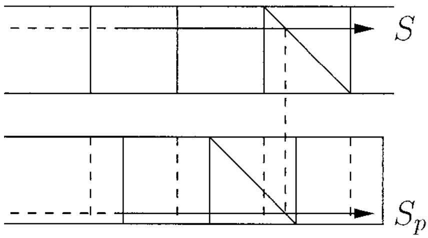

FIGURE 5

(we say condition (∗) holds), and

$$\binom{s+q+1}{p-1} = \cdots = \binom{s+q+u}{p-1} = 0; \qquad \binom{s+q+u+1}{p-1} = 1.$$

Again q always is such that $2^l \leqslant q-1 < 2^{l+1}$.

In the following two lemmas we give estimates for u. First, in Lemma 5.2, for Case a of same segmentation, and then, in Lemma 5.3, for Case b of different segmentation.

LEMMA 5.2. *In this lemma we always assume Case a. Especially this means, that for given* κ *and* σ, *for given* q *the parameter* s *with* $\kappa 2^{l-1} \leqslant s < (\kappa+1)\, 2^{l-1}$ *is uniquely determined, and conversely for given* s *the parameter* q *with* $\sigma 2^{l-1} \leqslant q < (\sigma+1)\, 2^{l-1}$ *is uniquely determined. Indeed,* $s+q+1$ *must be a multiple of* 2^{l-1}.

(a) *If* $S = S_p$ *then* $C_s = C_{S_p}$.

(b) *For given* S *and* S_p *we always have* $u \leqslant p-1$.

(c) *For given* r, s, p, q *with* $2^t \leqslant p-1 < 2^{t+1}$ *for some* t, *we have either*

(α) $u = p-1$ *and* $s+q+1 = A \cdot 2^{t+1}$ *for some positive integer* A *or*

(β) $u \leqslant p-1-2^t$.

Proof. (a) The fact that the characteristic chains must be the same is obvious.

(b) From the structure of the Pascal matrix (see figure) it is easily deduced that the length of the longest block of zeros in the kth row of the Pascal matrix is k (we start counting the row numbers with 0).

$$\left(\begin{array}{c|c}
\begin{array}{c|c} \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} & \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} \\ \hline 0 & \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} \end{array} & \begin{array}{c|c} \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} & \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} \\ \hline 0 & \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} \end{array} \\ \hline
0 & \begin{array}{c|c} \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} & \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} \\ \hline 0 & \begin{matrix}\ddots & C\\ 0 & \ddots\end{matrix} \end{array}
\end{array}\right)$$

The result trivially follows.

(c) Again considering the self-similar structure of the Pascal matrix we find that the second longest block of zeros in the kth row, where $2^t \leqslant k < 2^{t+1}$, is at most $k-2^t$. So $u = p-1$ or $u \leqslant p-1-2^t$.

The case $u = p-1$ only can occur if the string S_p terminates with the last column before a "big block of zeros" (see Fig. 6).

This means $s+q = A \cdot 2^{t+1} - 1$. The result follows. ∎

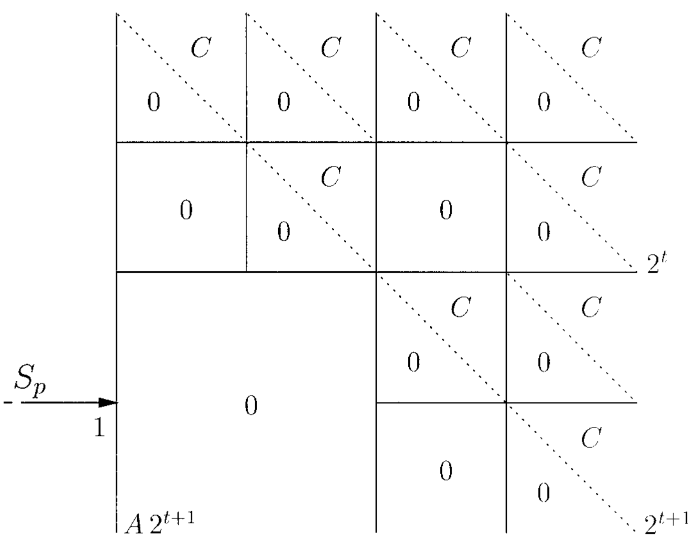

FIGURE 6

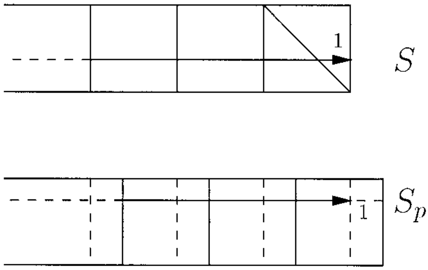

FIGURE 7

LEMMA 5.3. *In the following we always assume choice of the parameters such that we have different segmentation. We have*

(a) $u \leqslant 2^{l-1}$ *always.*

(b) C_S *and* C_{S_p} *are of the form* $0\,0 \ldots 0\,1$ *both of the form* $1\,1 \ldots 1\,1$ *both.*

Proof. (a) Let $S = S_p$. Since the last entry of C_S is one (and hence the last entry of S), also the last entry of C_{S_p} must be one (see Fig. 7). But then the number of the zeros following the last entry of S_p is at most 2^{l-1}.

(b) By the proof of (a) both C_S and C_{S_p} must terminate with one. Assume that any entry of C_S, resp. C_{S_p} is zero, then it is quite obvious that the corresponding entry of C_{S_p}, resp. C_S must be zero.

This means that C_S and C_{S_p} are of the form $0\,0 \ldots 0\,1$ both or of the form $1\,1 \ldots 1\,1$ both. ∎

In the subsequent we also will use the following simple fact (we omit the, easy proof).

LEMMA 5.4. *For* $s, l \geqslant 0$ *let* $D_l(s)$ *be a non-negative real. If for an absolute constant* c

$$\sum_{s=0}^{b-1} (D_l(s))^2 \leqslant c \cdot 2^l (a-l)$$

holds for all $l = 0, \ldots, a-1$, *then*

$$\sum_{s=0}^{b-1} \left(\sum_{l=0}^{a-1} D_l(s) \right)^2 \leqslant \bar{c} \cdot 2^a,$$

with an absolute constant $\bar{c}$.

We proceed with the proof of Proposition 5.1.

Remember that for given r, s, p, q we say that condition $(*)$ holds if the corresponding strings S and S_p are equal. Especially we have $L = M \leqslant q$ if $(*)$ does not hold. We have

$$\bar{\Sigma} \leqslant 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \Bigg(\sum_{\substack{p,q \\ (*)\text{ holds}}} \frac{2^L}{2^{p+q}} \Bigg)^2 + 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \Bigg(\sum_{\substack{p,q \\ (*)\text{ does not hold}}} \frac{2^L}{2^{p+q}} \Bigg)^2$$

$$=: 2 \Sigma_1 + 2 \Sigma_2.$$

We proceed with Σ_2:

$$\Sigma_2 \leqslant 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \Bigg(\sum_{q=1}^{4(s-r)} \sum_{p=s-r+1}^{s-r+q} \frac{2^q}{2^{p+q}} \Bigg)^2 + 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \Bigg(\sum_{q=4(s-r)+1}^{m} \sum_{p=s-r+1}^{s-r+q} \frac{2^L}{2^{p+q}} \Bigg)^2$$

(we denote the second sum by $2 \Sigma_3$)

$$\leqslant 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \left(\frac{4(s-r)}{2^{s-r}} \right)^2 + 2 \Sigma_3 \leqslant c \cdot m + 2 \Sigma_3,$$

with an absolute constant c.

Further, by using Lemma 5.1(a) we obtain

$$\Sigma_3 \leqslant \sum_{\substack{r,s \\ r<s}} \Bigg(\sum_{l=[\mathrm{ld}(s-r)]+2}^{[\mathrm{ld}(m)]} \sum_{q=2^l+1}^{2^{l+1}} \Bigg(\sum_{p=s-r}^{2^{l-1}} \frac{2^{q-2^{l-1}}}{2^{p+q}} + \sum_{p=2^{l-1}}^{s-r+q} \frac{2^q}{2^{p+q}} \Bigg) \Bigg)^2$$

$$\leqslant \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \Bigg(\sum_{l=[\mathrm{ld}(s-r)]+2}^{[\mathrm{ld}(m)]} \frac{2^{l+2}}{2^{2^{l-1}}} \Bigg)^2$$

$$\leqslant c \cdot m^2,$$

with an absolute constant c and $\mathrm{ld}(m) = \frac{\log m}{\log 2}$.

It remains to study Σ_1. We have

$$\Sigma_1 \leqslant 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \Bigg(\sum_{\substack{p,q \\ (*)\text{ holds,} \\ \text{case b}}} \frac{2^u}{2^p} \Bigg)^2 + 2 \sum_{r=1}^{m-2} \sum_{s=r+1}^{m-1} \Bigg(\sum_{\substack{p,q \\ (*)\text{ holds,} \\ \text{case a}}} \frac{2^u}{2^p} \Bigg)^2$$

$$=: 2 \Sigma_{11} + 2 \Sigma_{12}.$$

Further

$$\Sigma_{11} \leqslant \sum_{\substack{r,s\\ r<s}} \left(\sum_{l=1}^{[\mathrm{ld}(m)]+1} \sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1\\ (*)\text{ holds,}\\ \text{case b}}}^{s-r+q} \frac{2^u}{2^p} \right)^2.$$

From Lemma 5.3(b) we have that C_S and C_{S_p} are of the form 00...01 both or of the form 11...11 both.

If C_S and C_{S_p} are of the form 00...01 both, we have $p-1 \geqslant 2^l$ and therefore together with Lemma 5.3(a) we get

$$\frac{2^u}{2^p} \leqslant \frac{1}{2^{2^{l-1}}}.$$

Further by Lemma 5.1(b), for given r, s, and q, there are always at most $[q/2^{l-1}]+1 \leqslant 3$ values p for which $(*)$ holds. So

$$\sum_{\substack{r,s\\ r<s}} \left(\sum_{l=1}^{[\mathrm{ld}(m)]+1} \sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1\\ (*)\text{ holds,}\\ \text{case b}\\ C_S=C_{S_p}=0\ldots01}}^{s-r+q} \frac{2^u}{2^p} \right)^2 \leqslant \sum_{\substack{r,s\\ r<s}} \left(\sum_{l=1}^{\infty} 3\cdot 2^l \frac{1}{2^{2^{l-1}}} \right)^2 \leqslant c \cdot m^2,$$

with an absolute constant c.

Now we consider C_S and C_{S_p} of the form 11...11 both. Therefore we get either case (i) $p-1 \geqslant 2^{l+1}$ or case (ii) $p-1 < 2^l$.

In case (i) we get as above

$$\sum_{\substack{r,s\\ r<s}} \left(\sum_{l=1}^{[\mathrm{ld}(m)]+1} \sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1\\ (*)\text{ holds,}\\ \text{case b}\\ \text{case (i)}}}^{s-r+q} \frac{2^u}{2^p} \right)^2 \leqslant c \cdot m^2,$$

with an absolute constant c.

So we only have to consider case (ii), i.e., $p-1 < 2^l$. Since we always have $u < p$ (see the proof of Lemma 5.2(b)) we get (for short we write $(**)$ for property $(*)$, case (b) and case (ii))

$$\sum_{\substack{r,s\\ r<s}} \left(\sum_{l=1}^{[\mathrm{ld}(m)]+1} \sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1\\ (**)\text{ holds}}}^{s-r+q} \frac{2^u}{2^p} \right)^2 \leqslant \sum_{\substack{r,s\\ r<s}} \left(\sum_{l=1}^{[\mathrm{ld}(m)]+1} \sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1\\ (**)\text{ holds}}}^{2^l} 1 \right)^2$$

$$= \sum_{\substack{r,s\\ r<s}} \sum_{l_1,l_2} \left(\sum_{q_1=2^{l_1-1}}^{2^{l_1}-1} \sum_{\substack{p_1=s-r+1\\ (**)\text{ holds}}}^{2^{l_1}} 1 \right) \cdot \left(\sum_{q_2=2^{l_2-1}}^{2^{l_2}-1} \sum_{\substack{p_2=s-r+1\\ (**)\text{ holds}}}^{2^{l_2}} 1 \right).$$

We assume $l_1 < l_2$. ($l_1 \geqslant l_2$ is treated in absolutely the same way.) So we estimate the sum

$$\Sigma_4 := \sum_{\substack{l_1, l_2 \\ l_1 < l_2}} \sum_{\substack{r, s \\ r < s \\ s-r < 2^{l_1}}} \left(\sum_{q_1 = 2^{l_1-1}}^{2^{l_1}-1} \sum_{\substack{p_1 = s-r+1 \\ (**) \text{ holds}}}^{2^{l_1}} 1 \right) \cdot \left(\sum_{q_2 = 2^{l_1-1}}^{2^{l_2}-1} \sum_{\substack{p_2 = s-r+1 \\ (**) \text{ holds}}}^{2^{l_2}} 1 \right).$$

For $r = r_0 + r_1\, 2 + r_2\, 2^2 + \cdots$ and for $l \geqslant 1$ we define the number

$$\operatorname{per}(r, l) := \begin{cases} 0, & \text{if } \; r_0 = \cdots = r_{l-1} = 1, \\ \max\{i \leqslant l-1 : r_i = 0\} + 1, & \text{else.} \end{cases}$$

Consider now the first 2^{l+1} elements of the rth column of our Pascal matrix. Then it is easy to check, that this sequence has period $2^{\operatorname{per}(r, l)}$.

Since property $(**)$ holds we must have that $2^{\operatorname{per}(r+1, l_1)}$ is greater than the maximum period (related to l_1) of the $(p_1 - 1)$st row of the Pascal matrix. (The period of the kth row of the Pascal matrix for $2^t \leqslant k < 2^{t+1}$ equals 2^{t+1}.) Therefore, since $p_1 - 1 \geqslant s - r$, we must have

$$2^{\operatorname{per}(r+1, l_1)} \geqslant s - r.$$

By the same argument we must have

$$2^{\operatorname{per}(r+1, l_2)} \geqslant s - r.$$

(But since $l_2 > l_1$ it follows that $\operatorname{per}(r+1, l_2) \geqslant \operatorname{per}(r+1, l_1)$ and so the second condition is of no relevance.)

Moreover for given r, s and l_1 we have at most $2^{l_1 - \operatorname{per}(r+1, l_1)}$ pairs (q_1, p_1) such that property $(**)$ holds and for given r, s and l_2 we have at most $2^{l_2 - \operatorname{per}(r+1, l_2)}$ pairs (q_2, p_2) such that property $(**)$ holds. So we get

$$\begin{aligned} \Sigma_4 &\leqslant \sum_{\substack{l_1, l_2 \\ l_1 < l_2}} \sum_{s=1}^{m-1} \sum_{\substack{r = s - 2^{l_1} \\ 2^{\operatorname{per}(r+1, l_1)} \geqslant s - r}}^{s-1} \frac{2^{l_1 + l_2}}{2^{\operatorname{per}(r+1, l_1) + \operatorname{per}(r+1, l_2)}} \\ &\leqslant \sum_{\substack{l_1, l_2 \\ l_1 < l_2}} \sum_{u=0}^{[m/2^{l_1}]} \sum_{s = u\, 2^{l_1}}^{(u+1)\, 2^{l_1} - 1} \sum_{\substack{r = s - 2^{l_1} \\ 2^{\operatorname{per}(r+1, l_1)} \geqslant s - r}}^{s-1} \frac{2^{l_1 + l_2}}{2^{\operatorname{per}(r+1, l_1) + \operatorname{per}(r+1, l_2)}}. \end{aligned}$$

Consider now the binary digit expansion of $r+1$ and of $u\, 2^{l_1}$:

$$\begin{array}{lcllllllllll} r+1 & : & r_0 & r_1 & \ldots & r_{l_1-1} & r_{l_1} & r_{l_1+1} & \ldots & r_{l_2} & r_{l_2+1} & \ldots \\ u\, 2^{l_1} & : & 0 & 0 & \ldots & 0 & u_0 & u_1 & \ldots & u_{l_2-l_1} & u_{l_2-l_1+1} & \ldots \end{array}$$

Note that for $u\,2^{l_1} \leqslant s \leqslant (u+1)\,2^{l_1}-1$ we have $(u-1)\,2^{l_1}+1 \leqslant r+1 \leqslant (u+1)\,2^{l_1}-1$.

We say the number u has property (1), if there is at least one zero under the first l_2-l_1+1 digits of the binary digit expansion of u and of $u-1$ and we say u has property (2), if the first l_2-l_1+1 digits of the binary digit expansion of u or of $u-1$ consists only of ones. Then we have

$$\begin{aligned}\Sigma_4 \leqslant{} & \sum_{\substack{l_1, l_2\\ l_1<l_2}} \sum_{\substack{u=0\\ u \text{ has prop. (1)}}}^{[m/2^{l_1}]} \sum_{s=u\,2^{l_1}}^{(u+1)\,2^{l_1}-1} \sum_{\substack{r=s-2^{l_1}\\ 2^{\mathrm{per}(r+1,\,l_1)} \geqslant s-r}}^{s-1} \frac{2^{l_1+l_2}}{2^{\mathrm{per}(r+1,\,l_1)+\mathrm{per}(r+1,\,l_2)}}\\ & + \sum_{\substack{l_1, l_2\\ l_1<l_2}} \sum_{\substack{u=0\\ u \text{ has prop. (2)}}}^{[m/2^{l_1}]} \sum_{s=u\,2^{l_1}}^{(u+1)\,2^{l_1}-1} \sum_{\substack{r=s-2^{l_1}\\ 2^{\mathrm{per}(r+1,\,l_1)} \geqslant s-r}}^{s-1} \frac{2^{l_1+l_2}}{2^{\mathrm{per}(r+1,\,l_1)+\mathrm{per}(r+1,\,l_2)}}\\ =:{} & \sum{}^{(1)} + \sum{}^{(2)}.\end{aligned}$$

First we have

$$\begin{aligned}\sum{}^{(1)} \leqslant{} & \sum_{\substack{l_1, l_2\\ l_1<l_2}} \sum_{\substack{u=0\\ u \text{ has prop. (1)}}}^{[m/2^{l_1}]} \left(2^{l_2-\mathrm{per}(u\,2^{l_1},\,l_2)} + 2^{l_2-\mathrm{per}((u-1)\,2^{l_1},\,l_2)}\right)\\ & \times \sum_{s=u\,2^{l_1}}^{(u+1)\,2^{l_1}-1} \sum_{\substack{r=s-2^{l_1}\\ 2^{\mathrm{per}(r+1,\,l_1)} \geqslant s-r}}^{s-1} 2^{l_1-\mathrm{per}(r+1,\,l_1)}.\end{aligned}$$

If, for running s, we choose r such that $s-r$ is constant, then we get all possible combinations of the first l_1 digits in the binary digit expansion of $r+1$ at most once. Therefore little consideration yields

$$\sum_{s=u\,2^{l_1}}^{(u+1)\,2^{l_1}-1} \sum_{\substack{r=s-2^{l_1}\\ 2^{\mathrm{per}(r+1,\,l_1)} \geqslant s-r}}^{s-1} 2^{l_1-\mathrm{per}(r+1,\,l_1)} \leqslant 4^{l_1}$$

and hence

$$\sum{}^{(1)} \leqslant \sum_{\substack{l_1, l_2\\ l_1<l_2}} 4^{l_1} \sum_{u=0}^{[m/2^{l_1}]} \left(2^{l_2-\mathrm{per}(u\,2^{l_1},\,l_2)} + 2^{l_2-\mathrm{per}((u-1)\,2^{l_1},\,l_2)}\right).$$

Now we have

$$\sum_{u=0}^{[m/2^{l_1}]} (2^{l_2-\mathrm{per}(u\,2^{l_1},\,l_2)}) = \sum_{k=1}^{[m/2^{l_2}]} \sum_{u=(k-1)\,2^{l_2-l_1}}^{k2^{l_2-l_1}} 2^{l_2-\mathrm{per}(u\,2^{l_1},\,l_2)}$$
$$\leqslant \frac{m}{2^{l_2}} (2^{l_2-l_1-1}+2^{l_2-l_1-2}+\cdots+2^{l_2-l_2-1}2^{l_2-l_1})$$
$$\leqslant \frac{m}{2^{l_1}} (l_2-l_1+1).$$

In the same way we get

$$\sum_{u=0}^{[m/2^{l_1}]} 2^{l_2-\mathrm{per}((u-1)\,2^{l_1},\,l_2)} \leqslant \frac{m}{2^{l_1}} (l_2-l_1+1).$$

Therefore we have

$$\sum{}^{(1)} \leqslant 2m \sum_{\substack{l_1,\,l_2=1\\ l_1<l_2}}^{[\mathrm{ld}(m)]+1} 2^{l_1}(l_2-l_1+1) \leqslant c\cdot m^2,$$

with an absolute constant c.

Now consider $\sum^{(2)}$. We have

$$\sum{}^{(2)} \leqslant \sum_{\substack{l_1,\,l_2\\ l_1<l_2}} \sum_{\substack{u=0\\ u \text{ has prop. (2)}}}^{[m/2^{l_1}]} \sum_{s=u\,2^{l_1}}^{(u+1)\,2^{l_1}-1} \sum_{\substack{r=s-2^{l_1}\\ 2^{\mathrm{per}(r+1,\,l_1)}\geqslant s-r}}^{s-1} \frac{2^{l_1+l_2}}{4^{\mathrm{per}(r+1,\,l_1)}}$$
$$= \sum_{\substack{l_1,\,l_2\\ l_1<l_2}} 2^{l_2-l_1} \sum_{\substack{u=0\\ u \text{ has prop. (2)}}}^{[m/2^{l_1}]} \sum_{s=u\,2^{l_1}}^{(u+1)\,2^{l_1}-1} \sum_{\substack{r=s-2^{l_1}\\ 2^{\mathrm{per}(r+1,\,l_1)}\geqslant s-r}}^{s-1} 4^{l_1-\mathrm{per}(r+1,\,l_1)}.$$

As above we get

$$\sum_{s=u\,2^{l_1}}^{(u+1)\,2^{l_1}-1} \sum_{\substack{r=s-2^{l_1}\\ 2^{\mathrm{per}(r+1,\,l_1)}\geqslant s-r}}^{s-1} 4^{l_1-\mathrm{per}(r+1,\,l_1)} \leqslant l_1 4^{l_1}.$$

Now note that under $2^{l_2-l_1}$ consecutive numbers u there are at most two with property (2). Hence we have

$$\sum{}^{(2)} \leqslant \sum_{\substack{l_1,\,l_2=1\\ l_1<l_2}}^{[\mathrm{ld}(m)]+1} 2^{l_2-l_1}\, 2\,\frac{m}{2^{l_2}}\, l_1 4^{l_1} = 2m \sum_{\substack{l_1,\,l_2=1\\ l_1<l_2}}^{[\mathrm{ld}(m)]+1} l_1 2^{l_1} \leqslant c\cdot m^2\cdot \mathrm{ld}(m),$$

with an absolute constant c.

So, all together, we have shown

$$\Sigma_{11} > \leqslant c \cdot m^2 \cdot \mathrm{ld}(m),$$

with an absolute constant c.

Finally

$$\Sigma_{12} \leqslant \sum_{\substack{r,s \\ r<s}} \left(\sum_{l=1}^{[\mathrm{ld}(m)]+1} \sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1 \\ (*) \text{ holds,} \\ \text{case a}}}^{s-r+q} \frac{2^u}{2^p} \right)^2.$$

For given r and given l with $l \leqslant [\mathrm{ld}(m)]+1$ let us consider

$$\sum(r,l) := \sum_{s=r+1}^{m-1} \left(\sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1 \\ (*) \text{ holds,} \\ \text{case a}}}^{s-r+q} \frac{2^u}{2^p} \right)^2.$$

By Lemma 5.4 it suffices to show, that with an absolute constant c, for all r and l we have

$$\sum(r,l) \leqslant c \cdot 2^l(\mathrm{ld}(m)+2-l).$$

We consider first

$$\sum(r,l)_1 := \sum_{s=r+1}^{r+2^{l+1}} \left(\sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1 \\ (*) \text{ holds,} \\ \text{case a}}}^{s-r+q} \frac{2^u}{2^p} \right)^2.$$

For given r, s and l, the parameter q is uniquely determined in the range $2^{l-1} \leqslant q < 2^l$ (see formulation of Lemma 5.2), further by Lemma 5.2(a) we have $C_S = C_{S_p}$ and so by Lemma 5.1(b) for p we have at most three possibilities in the range $s-r+1 \leqslant p \leqslant s-r+q$. Further we use Lemma 5.2(b), i.e., $u \leqslant p-1$, so that

$$\sum(r,l)_1 \leqslant 2^{l+1} \cdot 9 \leqslant 18 \cdot 2^l(\mathrm{ld}(m)+2-l).$$

Finally

$$\sum(r,l)_2 := \sum_{s=r+2^{l}+1}^{m-1} \Bigg(\sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=s-r+1 \\ (*)\text{ holds,} \\ \text{case a}}}^{s-r+q} \frac{2^u}{2^p} \Bigg)^2,$$

is estimated as

$$\sum(r,l)_2 \leqslant \sum_{t=l+1}^{[\mathrm{ld}(m)]+1} \sum_{s=r+2^{t-1}+1}^{r+2^t} \Bigg(\sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=2^{t-1}+1 \\ s-r+1\leqslant p\leqslant s-r+q}}^{2^{t+1}} \frac{2^u}{2^p} \Bigg)^2.$$

Again, for every choice of r, l, t, s, and q, at most three p are of relevance. By Lemma 5.2(c) we have $u=p-1$ or $u\leqslant p-1-2^t$. So the choices of parameters in $\sum(r,l)_2$ for which $u<p-1$ give a value of at most

$$\sum_{t=l+1}^{[\mathrm{ld}(m)]+1} \sum_{s=r+2^{t-1}+1}^{r+2^t} \Bigg(3 \sum_{q=2^{l-1}}^{2^l-1} \frac{1}{2^{2^{t-1}}} \Bigg)^2 < c,$$

where c is an absolute constant.

The situation $u=p-1$ by Lemma 5.2(c) for a value p in the range $2^t+1\leqslant p\leqslant 2^{t+1}$ only can occur if $s+q+1=A2^{t+1}$ for some positive integer A. Further for every s, the only parameter q for which we have case a is determined by the property that $s+q+1$ is a multiple of 2^{l-1} (see Lemma 5.2), and again for every choice of r, l, t, s, and q at most three values p are of relevance (note that also $s-r+1\leqslant p\leqslant s-r+q$). So for the situation $u=p-1$ we have the conditions

$$r+2^{t-1}+1\leqslant s\leqslant r+2^t$$

$$s+q+1=\left(\left[\frac{s+1}{2^{l-1}}\right]+1\right)\cdot 2^{l-1}$$

(note that $2^{l-1}<2^{t-1}<s$), and

$$s+q+1=A2^{t+1},$$

for some positive integer A.

It is easily checked, that at most 2^l parameters s with uniquely corresponding q can satisfy these conditions. So the choices of parameters in $\sum(r,l)_2$, for which $u=p-1$ give a value of at most

$$\sum_{t=l+1}^{[\mathrm{ld}(m)]+1} \sum_{s=r+2^{t-1}+1}^{r+2^t} \Bigg(\sum_{q=2^{l-1}}^{2^l-1} \sum_{\substack{p=2^{t-1}+1 \\ s-r+1\leqslant p\leqslant s-r+q}}^{2^{t+1}} 1 \Bigg)^2 \leqslant ([\mathrm{ld}(m)]+1-l)\cdot 2^l\cdot 9,$$

and the result follows.

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